

# AN EXPLICIT STRUCTURE OF THE GRADED RING OF MODULAR FORMS WITH RESPECT TO CERTAIN CONGRUENCE GROUPS

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## 1. INTRODUCTION

We put  $\mathbb{M} = \{0, 1, 2, \dots\}$  and  $\mathbb{N} = \{1, 2, \dots\}$ . For each  $k \in \mathbb{M}$  and a congruence subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$ , let  $\mathcal{M}_k(\Gamma)$  be the space of all modular forms of weight  $k$  with respect to  $\Gamma$ . The main aim of this paper is to study the  $\mathbb{M}$ -graded ring

$$\mathcal{M}(\Gamma) = \bigoplus_{k \in \mathbb{M}} \mathcal{M}_k(\Gamma)$$

for certain  $\Gamma$ . The  $2\mathbb{M}$ -graded ring  $\mathcal{M}(\Gamma_0(N))$  is studied in [8], and we follow the methods of the paper. We regard  $\mathcal{M}(\Gamma)$  as a subring of  $\mathbb{C}[[q]]$ , where  $q = e^{2\pi\sqrt{-1}z}$  ( $z \in \mathcal{H}$ ), via the Fourier expansion.

We also study the modular forms of half-integer weight and the  $\frac{1}{2}\mathbb{M}$ -graded ring

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(\Gamma) = \bigoplus_{\kappa \in \frac{1}{2}\mathbb{M}} \mathcal{M}_{\kappa}(\Gamma)$$

for certain  $\Gamma$ .

## 2. PREPARATIONS—GENERAL THEORIES

2.1. **actions.** Put

$$\mathrm{GL}_2^+(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Q}, ad - bc > 0 \right\},$$

and define the action  $\mathrm{GL}_2^+(\mathbb{Q}) \curvearrowright \mathrm{GL}_2^+(\mathbb{Q})$  by

$$\gamma \triangleleft \alpha = \alpha^{-1} \gamma \alpha.$$

For example, we see

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleleft \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b/h \\ ch & d \end{pmatrix}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleleft \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a-ch & ah+b-ch^2-dh \\ c & d+ch \end{pmatrix}. \end{aligned}$$

Put  $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ , and define the action  $\mathrm{GL}_2^+(\mathbb{Q}) \curvearrowright \mathcal{H}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

In addition, put  $\mathcal{O}(\mathcal{H}) = \{f : \mathcal{H} \rightarrow \mathbb{C} \mid f : \text{holomorphic}\}$ , and define the action  $\mathcal{O}(\mathcal{H}) \curvearrowright \mathrm{GL}_2^+(\mathbb{Q})$  by

$$(f|_k \gamma)(z) = (cz+d)^{-k} f(\gamma z).$$

2.2. **graded rings.** We say a ring  $R$  is graded if it is decomposed into a direct sum of  $\mathbb{C}$ -vector spaces  $R = \bigoplus_{k \in \mathbb{M}} R_k$  such that  $R_k R_l \subset R_{k+l}$  for all  $k, l \in \mathbb{M}$ . In

this paper, we only deal with the case  $R_0 = \mathbb{C}$ . For example,  $\mathbb{C}$  is graded as  $\mathbb{C}_k = \{0\}$  for  $k \in \mathbb{N}$ . Moreover, for a graded ring  $R = \bigoplus_k R_k$  and  $n_1, \dots, n_r \in \mathbb{N}$ , we define  $S = R[X_1, \dots, X_m]^{[n_1, \dots, n_m]}$  to be a ring  $R[X_1, \dots, X_m]$  which is graded as  $X_i \in S_{n_i}$ . For  $R = \bigoplus_{k \in \mathbb{M}} R_k$ , we define a graded-ring  $R_{2\mathbb{M}} = \bigoplus_{k \in 2\mathbb{M}} R_k$ .

For given graded rings  $R = \bigoplus_{k \in \mathbb{M}} R_k$  and  $S = \bigoplus_{k \in \mathbb{M}} S_k$ , a ring homomorphism  $f : R \rightarrow S$  is said to be graded if  $f(R_k) \subset S_k$  for all  $k \in \mathbb{M}$ . In the sequel, every homomorphism is meant to be graded.

2.3. **modular forms.** For  $k \in \mathbb{M}$  and  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$  such that  $\Gamma(N, 1) \subset \Gamma$  for some  $N$ , put

$$\mathcal{M}_k(\Gamma) = \{f \in \mathcal{O}(\mathcal{H}) \mid \forall \gamma \in \Gamma, f|_k \gamma = f, f : \text{holomorphic at } \mathbb{Q} \cup \{\infty\}\},$$

$$\mathcal{M}(\Gamma) = \bigoplus_{k \in \mathbb{M}} \mathcal{M}_k(\Gamma).$$

Via the Fourier expansion, we regard

$$\mathcal{M}(\Gamma) \subset \mathbb{C}[[q]].$$

For  $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ , we may construct the isomorphism of  $\mathbb{C}$ -vector spaces

$$|_k \alpha : \mathcal{M}_k(\Gamma) \rightarrow \mathcal{M}_k(\Gamma \triangleleft \alpha),$$

and the isomorphism of graded rings

$$|\alpha = \bigoplus_k |_k \alpha : \mathcal{M}(\Gamma) \rightarrow \mathcal{M}(\Gamma \triangleleft \alpha).$$

In particular, for  $h \in \mathbb{N}$ , we write  $f^{(h)} = f|_0 \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ . Moreover, for  $f \in 1 + aq + \mathbb{C}[[q]]q^2$ , we define

$$f_{(h)} = \frac{1}{a}(f - f^{(h)}).$$

**2.4. congruence group.** For  $N \in \mathbb{N}$  and a subgroup  $G$  of  $\mathbb{Z}/N^\times$ , put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \natural_N(c) = 0 \right\}.$$

$$\Gamma(N, G) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid \natural_N(d) \in G \right\}.$$

where  $\natural_N$  is the natural map  $\mathbb{Z} \rightarrow \mathbb{Z}/N$ . We abbreviate  $\mathcal{M}_k(N, G) = \mathcal{M}_k(\Gamma(N, G))$  and  $\mathcal{M}_k(N) = \mathcal{M}_k(\Gamma_0(N))$ . For  $\chi : \mathbb{Z}/N^\times \rightarrow \mathbb{C}^\times$ , put

$$\mathcal{M}_k(N; \chi) = \left\{ f \in \mathcal{M}_k(N, 1) \mid f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

Note

$$\Gamma_0(N) \triangleleft \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \supset \Gamma_0(hN)$$

and  $\mathcal{M}(N; \chi)^{\langle h \rangle} \subset \mathcal{M}(hN; \chi)$ .

**Lemma 1.** *We have*

$$\mathcal{M}_k(N, G) = \bigoplus_{\chi(G)=\{1\}} \mathcal{M}_k(N; \chi).$$

*Proof.* Note  $\mathcal{M}_k(N, 1) = \bigoplus_{\chi} \mathcal{M}_k(N; \chi)$  (cf. [2, §4.3] or [9, Proposition 9.2]).

Let  $f \in \mathcal{M}_k(N, G)$ . We can write  $f = \sum_{\chi} f_{\chi}$  with  $f_{\chi} \in \mathcal{M}_k(N; \chi)$ . Then, for each  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ , we see

$$\sum_{\chi} f_{\chi} = \sum_{\chi} f_{\chi}|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{\chi} \chi(d) f_{\chi}.$$

If  $\chi(G) \neq \{1\}$ , then we can choose integers  $a, b, c, d$  such that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N, G)$  and  $\chi(d) \neq 1$ , therefore  $f_{\chi} = 0$ . That is,  $f \in \bigoplus_{\chi(G)=\{1\}} \mathcal{M}_k(N; \chi)$ .

Conversely, if  $\chi(G) = \{1\}$  then we easily see  $\mathcal{M}_k(N, G) \supset \mathcal{M}_k(N; \chi)$ .  $\square$

If  $\chi : \mathbb{Z}/N^\times \not\rightarrow \mathbb{R}^\times$ , we abbreviate  $\&\mathcal{M}_k(N; \chi) = \mathcal{M}_k(N; \chi) \oplus \mathcal{M}_k(N; \overline{\chi})$ .

**2.5. Eisenstein series.** For  $k \in \mathbb{N}$  and  $n \in \mathbb{N}$ , define  $\sigma_k(n) = \sum_{d|n} d^k$ . For  $k \in 2\mathbb{N}$ , let  $B_k$  be the  $k$ -th Bernoulli number and

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \in \mathbb{N}} \sigma_{k-1}(n) q^n,$$

then we see  $E_k \in \mathcal{M}_k(1)$  if  $k \geq 4$ . For  $N > 1$ , we put

$$C_N = \frac{1}{N-1} (N E_2^{\langle N \rangle} - E_2) \in \mathcal{M}_2(N).$$

We see

$$E_4 = 1 + 240 \sum_{n \in \mathbb{N}} \sigma_3(n) q^n,$$

$$E_6 = 1 - 504 \sum_{n \in \mathbb{N}} \sigma_5(n) q^n,$$

$$C_p = 1 + \frac{24}{p-1} \sum_{n \in \mathbb{N}} (\sigma_1 * \mathbf{1}_p)(n) q^n \quad \text{for prime } p,$$

$$C_4 = 1 + 8 \sum_{n \in \mathbb{N}} (\sigma_1 * \mathbf{1}_2)(n) (q^n + 2q^{2n}),$$

where  $(\sigma_k * \rho)(n) = \sum_{d|n} \rho(d) d^k$ . Remark

$$C_N^{\langle h \rangle} = \frac{1}{N-1} (N E_2^{\langle Nh \rangle} - E_2^{\langle h \rangle}) = \frac{1}{(N-1)h} ((Nh-1)C_{Nh} - (h-1)C_h).$$

If  $\chi : \mathbb{Z}/N^\times \rightarrow \mathbb{C}^\times$  is primitive and  $k \in \mathbb{N}$  such that  $\chi(-1) = (-1)^k$ , let  $B_{k,\chi}$  be the generalized Bernoulli number and

$$\begin{aligned} f_{k;\chi} &= 1 - \frac{2k}{B_{k,\chi}} \sum_{n \in \mathbb{N}} (\sigma_{k-1} * \chi)(n) q^n \in \mathcal{M}_k(N; \chi), \\ g_{k;\chi} &= \sum_{n \in \mathbb{N}} \left( \sum_{d|n} \chi\left(\frac{n}{d}\right) d^{k-1} \right) q^n \in \mathcal{M}_k(N; \chi) \quad \text{for } k \geq 2. \end{aligned}$$

Moreover if  $\psi : \mathbb{Z}/M^\times \rightarrow \mathbb{C}^\times$  is primitive and  $\chi\psi(-1) = (-1)^k$ , then put

$$g_{k;\chi,\psi} = \sum_{n \in \mathbb{N}} \left( \sum_{d|n} \psi(d) \chi\left(\frac{n}{d}\right) d^{k-1} \right) q^n \in \mathcal{M}_k(NM; \chi\psi).$$

See [2, §4] or [9, §5.3] for further details. We abbreviate  $f_\chi = f_{1;\chi}$ .

## 3. PREPARATIONS—DATAS

3.1. **charcter and modular forms of weight 1.** For  $n \in \mathbb{N}$  we put

$$1^{1/n} = e^{2\pi\sqrt{-1}/n}.$$

Remark

$$\begin{aligned} x = 1^{1/2} &\implies x + 1 = 0, \\ x = 1^{1/4} &\implies x^2 + 1 = 0, \\ x = 1^{1/6} &\implies x^2 - x + 1 = 0, \\ x = 1^{1/10} &\implies x^4 - x^3 + x^2 - x + 1 = 0. \end{aligned}$$

Let  $\rho_3 : \mathbb{Z}/3^\times \rightarrow \mathbb{R}^\times$  such that  $\rho_3(-1) = 1^{1/2}$ , then

$$f_{\rho_3} = 1 + 6 \sum_{n \in \mathbb{N} \setminus (3\mathbb{M}+2)} (\sigma_0 * \rho_3)(n)q^n.$$

Let  $\rho_4 : \mathbb{Z}/4^\times \rightarrow \mathbb{R}^\times$  such that  $\rho_4(-1) = 1^{1/2}$ , then

$$f_{\rho_4} = 1 + 4 \sum_{n \in \mathbb{N} \setminus (4\mathbb{M}+3)} (\sigma_0 * \rho_4)(n)q^n.$$

Let  $\chi_5 : \mathbb{Z}/5^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_5(2) = 1^{1/4}$ , and  $\rho_5 = \chi_5^2$ , then

$$f_{\chi_5} = 1 + (3 - 1^{1/4}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_5)(n)q^n.$$

Let  $\chi_7 : \mathbb{Z}/7^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_7(3) = 1^{1/6}$ , and  $\rho_7 = \chi_7^3$ , then

$$\begin{aligned} f_{\chi_7} &= 1 + (3 - 2 \cdot 1^{1/6}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_7)(n)q^n, \\ f_{\rho_7} &= 1 + 2 \sum_{n \in \mathbb{N} \setminus (7\mathbb{M} + \{3,5,6\})} (\sigma_0 * \rho_7)(n)q^n. \end{aligned}$$

Let  $\rho_8 : \mathbb{Z}/8^\times \rightarrow \mathbb{R}^\times$  such that  $\rho_8(5) = \rho_8(-1) = 1^{1/2}$ , then

$$f_{\rho_8} = 1 + 2 \sum_{n \in \mathbb{N} \setminus (8\mathbb{M} + \{5,7\})} (\sigma_0 * \rho_8)(n)q^n.$$

Let  $\chi_9 : \mathbb{Z}/9^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_9(2) = 1^{1/6}$ , then  $\chi_9^3 = \rho_3$ ,

$$f_{\chi_9} = 1 + (2 - 1^{1/6}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_9)(n)q^n.$$

Let  $\chi_{11} : \mathbb{Z}/11^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{11}(2) = 1^{1/10}$ , and  $\rho_{11} = \chi_{11}^5$ , then

$$f_{\chi_{11}} = 1 + (2 - 2 \cdot 1^{1/10} + 2 \cdot 1^{2/10} - 1^{3/10}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_{11})(n)q^n,$$

$$f_{\chi_{11}^3} = 1 + (1 - 1^{1/10} - 1^{2/10} - 1^{3/10}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_{11}^3)(n)q^n,$$

$$f_{\rho_{11}} = 1 + 2 \sum_{n \in \mathbb{N} \setminus (11\mathbb{M} + \{2,6,7,8,10\})} (\sigma_0 * \rho_{11})(n)q^n.$$

Let  $\chi_{13} : \mathbb{Z}/13^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{13}(2) = 1^{1/12}$  and  $\rho_{13} = \chi_{13}^6$ .

Let  $\chi_{16} : \mathbb{Z}/16^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{16}(-1) = 1^{1/2}$ ,  $\chi_{16}(5) = 1^{1/4}$ , then  $\chi_{16}^2 = \rho_4\rho_8$ ,

$$f_{\chi_{16}} = 1 + (1 - 1^{1/4}) \sum_{n \in \mathbb{N}} (\sigma_0 * \chi_{16})(n)q^n.$$

Let  $\chi_{17} : \mathbb{Z}/17^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{17}(2) = 1^{1/16}$  and  $\rho_{17} = \chi_{17}^8$ .

Let  $\chi_{19} : \mathbb{Z}/19^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{19}(2) = 1^{1/18}$  and  $\rho_{19} = \chi_{19}^9$ .

Let  $\chi_{23} : \mathbb{Z}/23^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_{23}(5) = 1^{1/22}$  and  $\rho_{23} = \chi_{23}^{11}$ .

$$\alpha_{23} = \sum_{m,n \in \mathbb{Z}} (q^{m^2+mn+6n^2} - q^{2m^2+mn+3n^2}).$$

(cf. [3])

### 3.2. decomposition.

$$\begin{aligned} \mathcal{M}_k(3, 1) &= \begin{cases} \mathcal{M}_k(3) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(3; \rho_3) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(4, 1) &= \begin{cases} \mathcal{M}_k(4) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(4; \rho_4) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(5, 1) &= \begin{cases} \mathcal{M}_k(5) \oplus \mathcal{M}_k(5; \rho_5) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(5; \chi_5) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(6, 1) &= \begin{cases} \mathcal{M}_k(6) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(6; \rho_3) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(7, 1) &= \begin{cases} \mathcal{M}_k(7) \oplus \&\mathcal{M}_k(7; \chi_7^2) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(7; \chi_7) \oplus \mathcal{M}_k(7; \rho_7) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(8, 1) &= \begin{cases} \mathcal{M}_k(8) \oplus \mathcal{M}_k(8; \rho_4 \rho_8) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(8; \rho_4) \oplus \mathcal{M}_k(8; \rho_8) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(9, 1) &= \begin{cases} \mathcal{M}_k(9) \oplus \&\mathcal{M}_k(9; \chi_9^2) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(9; \chi_9) \oplus \mathcal{M}_k(9; \rho_3) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(10, 1) &= \begin{cases} \mathcal{M}_k(10) \oplus \mathcal{M}_k(10; \rho_5) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(10; \chi_5) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(11, 1) &= \begin{cases} \mathcal{M}_k(11) \oplus \&\mathcal{M}_k(11; \chi_{11}^2) \oplus \&\mathcal{M}_k(11; \chi_{11}^4) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(11; \chi_{11}) \oplus \&\mathcal{M}_k(11; \chi_{11}^3) \oplus \mathcal{M}_k(11; \rho_{11}) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(12, 1) &= \begin{cases} \mathcal{M}_k(12) \oplus \mathcal{M}_k(12; \rho_3 \rho_4) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(12; \rho_3) \oplus \mathcal{M}_k(12; \rho_4) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(13, 1) &= \begin{cases} \mathcal{M}_k(13) \oplus \&\mathcal{M}_k(13; \chi_{13}^2) \oplus \&\mathcal{M}_k(13; \chi_{13}^4) \oplus \mathcal{M}_k(13; \rho_{13}) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(13; \chi_{13}) \oplus \&\mathcal{M}_k(13; \chi_{13}^3) \oplus \&\mathcal{M}_k(13; \chi_{13}^5) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(13, \langle 3 \rangle) &= \begin{cases} \mathcal{M}_k(13) \oplus \mathcal{M}_k(13; \rho_{13}) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(13; \chi_{13}^3) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(14, 1) &= \begin{cases} \mathcal{M}_k(14) \oplus \&\mathcal{M}_k(14; \chi_7^2) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(14; \chi_7) \oplus \mathcal{M}_k(14; \rho_7) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(15, 1) &= \begin{cases} \mathcal{M}_k(15) \oplus \mathcal{M}_k(15; \rho_5) \oplus \&\mathcal{M}_k(15; \rho_3 \chi_{15}) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(15; \rho_3) \oplus \&\mathcal{M}_k(15; \chi_5) \oplus \mathcal{M}_k(15; \rho_3 \rho_5) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\ \mathcal{M}_k(16, 1) &= \begin{cases} \mathcal{M}_k(16) \oplus \mathcal{M}_k(16; \rho_4 \rho_8) \oplus \&\mathcal{M}_k(16; \rho_4 \chi_{16}) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(16; \rho_4) \oplus \mathcal{M}_k(16; \rho_8) \oplus \&\mathcal{M}_k(16; \chi_{16}) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \end{aligned}$$

$$\begin{aligned}
\mathcal{M}_k(16, \langle 9 \rangle) &= \begin{cases} \mathcal{M}_k(16) \oplus \mathcal{M}_k(16; \rho_4 \rho_8) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(16; \rho_4) \oplus \mathcal{M}_k(16; \rho_8) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\
\mathcal{M}_k(18, \langle 7 \rangle) &= \begin{cases} \mathcal{M}_k(18) & \text{if } k \in 2\mathbb{M}, \\ \mathcal{M}_k(18; \rho_3) & \text{if } k \in 2\mathbb{M} + 1, \end{cases} \\
\mathcal{M}_k(25, \langle 6 \rangle) &= \begin{cases} \mathcal{M}_k(25) \oplus \mathcal{M}_k(25, \rho_5) & \text{if } k \in 2\mathbb{M}, \\ \&\mathcal{M}_k(25, \chi_5) & \text{if } k \in 2\mathbb{M} + 1, \end{cases}
\end{aligned}$$

**3.3. dimension.** Dimension formulas are well-known (cf. [2, Theorem 3.5.1], [9, §6.3] or [7]). For  $k \in 2\mathbb{M}$ , we have

$$\dim \mathcal{M}_k(1) = \left\lfloor \frac{k}{12} \right\rfloor + 1 - \begin{cases} 1 & \text{if } k \in 12\mathbb{M} + 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim \mathcal{M}_k(2) = \left\lfloor \frac{k}{4} \right\rfloor + 1,$$

for  $k \in \mathbb{M}$

$$\begin{aligned}
\dim \mathcal{M}_k(3, 1) &= \left\lfloor \frac{k}{3} \right\rfloor + 1, \\
\dim \mathcal{M}_k(4, 1) &= \left\lfloor \frac{k}{2} \right\rfloor + 1, \\
\dim \mathcal{M}_k(5, 1) &= k + 1, \\
\dim \mathcal{M}_k(6, 1) &= k + 1, \\
\dim \mathcal{M}_k(7, 1) &= 2k + 1, \\
\dim \mathcal{M}_k(8, 1) &= 2k + 1, \\
\dim \mathcal{M}_k(9, 1) &= 3k + 1, \\
\dim \mathcal{M}_k(10, 1) &= 3k + 1, \\
\dim \mathcal{M}_k(12, 1) &= 4k + 1, \\
\dim \mathcal{M}_k(13, \langle 3 \rangle) &= k + 4 \left\lfloor \frac{k}{3} \right\rfloor + 1, \\
\dim \mathcal{M}_k(16, \langle 9 \rangle) &= 4k + 1, \\
\dim \mathcal{M}_k(18, \langle 7 \rangle) &= 3k + 1, \\
\dim \mathcal{M}_k(25, \langle 6 \rangle) &= 5k + 1,
\end{aligned}$$

and for  $k \in \mathbb{N}$

$$\begin{aligned}
\dim \mathcal{M}_k(11, 1) &= 5k, \\
\dim \mathcal{M}_k(13, 1) &= 7k - 1, \\
\dim \mathcal{M}_k(14, 1) &= 6k, \\
\dim \mathcal{M}_k(15, 1) &= 8k, \\
\dim \mathcal{M}_k(16, 1) &= 8k - 1.
\end{aligned}$$

## 4. GENERATORS

4.1. **the case**  $N = 1$ . Put

$$\alpha_1 = \frac{1}{12^3}(\mathbf{E}_4^3 - \mathbf{E}_6^2),$$

then we see  $\alpha_1 \in q + \mathbb{Z}[[q]]q^2$  and

$$\left\{ \begin{array}{l} \mathcal{M}_{12k}(1) = \bigoplus_{i=0}^k \mathbb{C}\mathbf{E}_4^{3(k-i)} \alpha_1^i, \\ \mathcal{M}_{12k+4}(1) = \mathcal{M}_{12k}(1)\mathbf{E}_4, \\ \mathcal{M}_{12k+6}(1) = \mathcal{M}_{12k}(1)\mathbf{E}_6, \\ \mathcal{M}_{12k+8}(1) = \mathcal{M}_{12k}(1)\mathbf{E}_4^2, \\ \mathcal{M}_{12k+10}(1) = \mathcal{M}_{12k}(1)\mathbf{E}_4\mathbf{E}_6, \\ \mathcal{M}_{12k+14}(1) = \mathcal{M}_{12k}(1)\mathbf{E}_4^2\mathbf{E}_6. \end{array} \right.$$

Thus, we get the natural surjective homomorphism

$$\mathbb{C}[\mathbf{E}_4, \mathbf{E}_6]^{[4,6]} \twoheadrightarrow \mathcal{M}(1).$$

4.2. **the case**  $N = 2$ . Put  $\alpha_2 = \mathbf{E}_{4\langle 2 \rangle}$ , then we get

$$\left\{ \begin{array}{l} \mathcal{M}_{4k}(2) = \bigoplus_{i=0}^k \mathbb{C}\mathbf{C}_2^{2(k-i)} \alpha_2^i, \\ \mathcal{M}_{4k+2}(2) = \mathcal{M}_{4k}(2)\mathbf{C}_2, \end{array} \right.$$

and

$$\mathbb{C}[\mathbf{C}_2, \alpha_2]^{[2,4]} \twoheadrightarrow \mathcal{M}(2).$$

4.3. **the case**  $N = 3$ . We get

$$\left\{ \begin{array}{l} \mathcal{M}_{3k}(3, 1) = \bigoplus_{i=0}^k \mathbb{C}\mathbf{f}_{\rho_3}^{3k-i} \mathbf{g}_{3;\rho_3}^i, \\ \mathcal{M}_{3k+1}(3, 1) = \mathcal{M}_{3k}(3, 1)\mathbf{f}_{\rho_3}, \\ \mathcal{M}_{3k+2}(3, 1) = \mathcal{M}_{3k}(3, 1)\mathbf{f}_{\rho_3}^2, \end{array} \right.$$

and

$$\mathbb{C}[\mathbf{f}_{\rho_3}, \mathbf{g}_{3;\rho_3}]^{[1,3]} \twoheadrightarrow \mathcal{M}(3, 1).$$

Note  $\mathbf{C}_3 = \mathbf{f}_{\rho_3}^2$  since  $\mathbf{C}_3 - \mathbf{f}_{\rho_3}^2 \in \mathcal{M}_2(3) \cap \mathbb{C}[[q]]q = \{0\}$ , and  $\mathbf{E}_{4\langle 3 \rangle} = \mathbf{f}_{\rho_3}\mathbf{g}_{3;\rho_3}$  similarly.

4.4. **the case**  $N = 4$ . Put

$$\alpha_4 = \mathbf{C}_{2\langle 2 \rangle} = \sum_{n \in 2\mathbb{M}+1} \sigma_1(n)q^n,$$

then we get

$$\mathbb{C}[\mathbf{f}_{\rho_4}, \alpha_4]^{[1,2]} \twoheadrightarrow \mathcal{M}(4, 1).$$

Note  $\mathbf{C}_4 = \mathbf{f}_{\rho_4}^2$ .



4.5. **the case**  $N = 5$ . Put

$$F_5 = \frac{1}{2}(f_{\chi_5} + \overline{f_{\chi_5}}) = 1 + \sum_{n \in \mathbb{N}} (\sigma_0 * (3\text{Re} + \text{Im})\chi_5)(n)q^n,$$

$$G_5 = \frac{1^{1/4}}{2}(f_{\chi_5} - \overline{f_{\chi_5}}) = \sum_{n \in \mathbb{N}} (\sigma_0 * (\text{Re} - 3\text{Im})\chi_5)(n)q^n,$$

then we get

$$\mathcal{M}_k(5, 1) = \bigoplus_{i=0}^k \mathbb{C} F_5^{k-i} G_5^i,$$

and

$$\mathbb{C}[F_5, G_5]^{[1,1]} \twoheadrightarrow \mathcal{M}(5, 1),$$

$$\mathbb{C}[f_{\chi_5}, \overline{f_{\chi_5}}]^{[1,1]} \twoheadrightarrow \mathcal{M}(5, 1).$$

Note  $C_5 = f_{\chi_5} \overline{f_{\chi_5}}$ .

4.6. **the case**  $N = 6$ . Put  $\alpha_6 = f_{\rho_3 \langle 2 \rangle}$ , then we get

$$\mathbb{C}[f_{\rho_3}, \alpha_6]^{[1,1]} \twoheadrightarrow \mathcal{M}(6, 1).$$

4.7. **the case**  $N = 7$ . Put

$$\alpha_7 = \frac{1}{3}(1^{1/6}f_{\chi_7} + \overline{1^{1/6}f_{\chi_7}} - f_{\rho_7}) = \sum_{n \in \mathbb{N}} (\sigma_0 * \frac{1}{3}((5\text{Sre} + \text{Sim})\chi_7 - 2\rho_7))(n)q^n,$$

$$\beta_7 = \frac{1}{3}(2f_{\rho_7} - f_{\chi_7} - \overline{f_{\chi_7}}) = \sum_{n \in \mathbb{N}} (\sigma_0 * \frac{1}{3}(4\rho_7 - (4\text{Sre} + 5\text{Sim})\chi_7))(n)q^n,$$

where  $\text{Sre}, \text{Sim} : \mathbb{C} \rightarrow \mathbb{R}$  such that  $\text{Sre}(x + y1^{1/6}) = x$  and  $\text{Sim}(x + y1^{1/6}) = y$ , then we get

$$\mathcal{M}_k(7, 1) = \bigoplus_{i=0}^k \mathbb{C} f_{\rho_7}^{k-i} \alpha_7^i \oplus \bigoplus_{i=1}^k \mathbb{C} \alpha_7^{k-i} \beta_7^i,$$

and

$$\mathbb{C}[f_{\rho_7}, \alpha_7, \beta_7]^{[1,1,1]} \twoheadrightarrow \mathcal{M}(7, 1),$$

$$\mathbb{C}[f_{\rho_7}, f_{\rho_7}, \overline{f_{\rho_7}}]^{[1,1,1]} \twoheadrightarrow \mathcal{M}(7, 1).$$

Remark  $\alpha_7 \in q + \mathbb{Z}[[q]]q^2$  since  $\frac{1}{3}((5\text{Sre} + \text{Sim})\chi_7 - 2\rho_7)(d) \in \{0, \pm 1, \pm 2\}$  for  $d \in \mathbb{N}$ .  
Note  $C_7 = f_{\rho_7}^2 = f_{\chi_7} \overline{f_{\chi_7}}$ .

4.8. **the case**  $N = 8$ . Put

$$\alpha_8 = f_{\rho_4 \langle 2 \rangle} = \sum_{n \in 4\mathbb{M}+1} (\sigma_0 * \rho_4)(n)q^n,$$

$$\beta_8 = \frac{1}{4}(f_{\rho_4} + f_{\rho_4}^{(2)} - 2f_{\rho_8}),$$

then we get

$$\mathbb{C}[f_{\rho_4}, \alpha_8, \beta_8]^{[1,1,1]} \twoheadrightarrow \mathcal{M}(8, 1).$$

4.9. **the case**  $N = 9$ . Put

$$\begin{aligned} F_9 &= 1^{1/6} f_{\chi_9} + \overline{1^{1/6} f_{\chi_9}} = 1 + 3 \sum_{n \in \mathbb{N}} (\sigma_0 * \text{Sre} \chi_9)(n) q^n, \\ \alpha_9 &= f_{\rho_3 \langle 3 \rangle} = \sum_{n \in 3\mathbb{M}+1} (\sigma_0 * \rho_3)(n) q^n, \\ \beta_9 &= \frac{1}{3} (F_9 - f_{\rho_3}^{(3)}) - \alpha_9, \\ \gamma_9 &= \frac{1}{3} (2F_9 - f_{\chi_9} - \overline{f_{\chi_9}}) - \alpha_9, \end{aligned}$$

then we get

$$\mathbb{C}[f_{\rho_3}^{(3)}, \alpha_9, \beta_9, \gamma_9]^{[1,1,1,1]} \twoheadrightarrow \mathcal{M}(9, 1).$$

4.10. **the case**  $N = 10$ . Put

$$\gamma_{10} = \frac{1}{10} (F_5^{(2)} - F_5 + 3G_5 + 7G_5^{(2)}),$$

then we get

$$\mathbb{C}[F_5, G_5, G_5^{(2)}, \gamma_{10}]^{[1,1,1,1]} \twoheadrightarrow \mathcal{M}(10, 1).$$

4.11. **the case**  $N = 11$ . Put

$$\begin{aligned} \alpha_{11} &= \frac{1}{5} (2f_{\rho_{11}}^2 - f_{\chi_{11}} \overline{f_{\chi_{11}}} - f_{\chi_{11}^3} \overline{f_{\chi_{11}^3}}), \\ \beta_{11} &= \frac{1}{5} ((1^{4/10} + 1^{1/10})(f_{\chi_{11}} \overline{f_{\chi_{11}^3}} - \overline{f_{\chi_{11}}} f_{\chi_{11}^3}^2) + (1^{3/10} + 1^{2/10})(f_{\chi_{11}^3} f_{\chi_{11}}^2 - \overline{f_{\chi_{11}^3}} \overline{f_{\chi_{11}}^2})), \\ \gamma_{11} &= \frac{1}{11} (f_{\rho_{11}} \beta_{11} - \alpha_{11}^2), \\ \varpi_{11} &= \frac{1}{11} (f_{\rho_{11}} \gamma_{11} - \alpha_{11} \beta_{11}), \end{aligned}$$

then we get

$$\left\{ \begin{array}{l} \mathcal{M}_1(11; \rho_{11}) = \mathbb{C} f_{\rho_{11}}, \\ \mathcal{M}_{5k+2}(11; \rho_{11}^k) = \mathcal{M}_{5k+1}(11; \rho_{11}^{k+1}) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^k \alpha_{11}, \\ \mathcal{M}_{5k+3}(11; \rho_{11}^{k+1}) = \mathcal{M}_{5k+2}(11; \rho_{11}^k) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^k \beta_{11}, \\ \mathcal{M}_{5k+4}(11; \rho_{11}^k) = \mathcal{M}_{5k+3}(11; \rho_{11}^{k+1}) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^k \gamma_{11}, \\ \mathcal{M}_{5k+5}(11; \rho_{11}^{k+1}) = \mathcal{M}_{5k+4}(11; \rho_{11}^k) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^{k+1}, \\ \mathcal{M}_{5k+6}(11; \rho_{11}^k) = \mathcal{M}_{5k+5}(11; \rho_{11}^{k+1}) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^k \alpha_{11} \gamma_{11}, \end{array} \right.$$

$$\mathcal{M}_{5k+1}(11; \rho_{11}^k \chi_{11}) = \mathcal{M}_{5k}(11; \rho_{11}^{k+1} \chi_{11}) f_{\rho_{11}} \oplus \mathbb{C} \varpi_{11}^k f_{\chi_{11}},$$

etc. . . , for  $k \in \mathbb{M}$ . We get

$$\mathbb{C}[f_{\chi_{11}}, f_{\chi_{11}^3}, f_{\rho_{11}}, \overline{f_{\chi_{11}^3}}, \overline{f_{\chi_{11}}}]^{[1,1,1,1,1]} \twoheadrightarrow \mathcal{M}(11, 1).$$

4.12. **the case**  $N = 12$ . Put

$$\begin{aligned} \alpha_{12} &= \alpha_6 + \alpha_6^{(2)} = \sum_{n \in (2\mathbb{M}+1) \setminus (3\mathbb{M}+2)} (\sigma_0 * \rho_3)(n) q^n, \\ \delta_{12} &= \frac{1}{6} (f_{\rho_3} + f_{\rho_3}^{(2)} + f_{\rho_3}^{(4)}) - \frac{1}{4} (f_{\rho_4} + f_{\rho_4}^{(3)}), \\ \gamma_{12} &= \frac{1}{4} (f_{\rho_4}^{(3)} - f_{\rho_3}^{(4)}) - \frac{1}{2} \delta_{12}, \end{aligned}$$

then we get

$$\mathbb{C}[f_{\rho_3}^{(4)}, \alpha_{12}, \alpha_6^{(2)}, \gamma_{12}, \delta_{12}]^{[1,1,1,1,1]} \twoheadrightarrow \mathcal{M}(12, 1).$$

4.13. **the case**  $N = 13$ . Put

$$\begin{aligned} F_{13} &= \frac{1}{2}(f_{\chi_{13}^3} + \overline{f_{\chi_{13}^3}}), & G_{13} &= \frac{1^{1/4}}{2}(f_{\chi_{13}^3} - \overline{f_{\chi_{13}^3}}), \\ x_0 &= \frac{1}{2}(f_{3;\chi_{13}^3} + \overline{f_{3;\chi_{13}^3}}), & x_1 &= \frac{1^{1/4}}{2}(f_{3;\chi_{13}^3} - \overline{f_{3;\chi_{13}^3}}), \\ y_1 &= \frac{1}{2}(g_{3;\chi_{13}^3} + \overline{g_{3;\chi_{13}^3}}), & y_2 &= \frac{1^{1/4}}{2}(g_{3;\chi_{13}^3} - \overline{g_{3;\chi_{13}^3}}), \end{aligned}$$

$$\begin{aligned} \delta_{13} &= \frac{1}{108}(-3F_{13}^3 + 24F_{13}^2G_{13} + 21F_{13}G_{13}^2 + 18G_{13}^3 + 3x_0 - 18x_1 - 18y_1 - 35y_2), \\ \epsilon_{13} &= \frac{1}{216}(21F_{13}^3 - 45F_{13}^2G_{13} + 111F_{13}G_{13}^2 - 15G_{13}^3 - 21x_0 + 15x_1 - 17y_1 + 161y_2), \\ \zeta_{13} &= \frac{1}{108}(-6F_{13}^3 - 21F_{13}^2G_{13} - 54F_{13}G_{13}^2 - 75G_{13}^3 + 6x_0 + 75x_1 + 53y_1 + 68y_2), \\ \eta_{13} &= \frac{1}{216}(-63F_{13}^3 + 273F_{13}^2G_{13} - 357F_{13}G_{13}^2 + 483G_{13}^3 + 63x_0 - 267x_1 - 127y_1 - 975y_2), \end{aligned}$$

then we get

$$\mathbb{C}[F_{13}, G_{13}, \delta_{13}, \epsilon_{13}, \zeta_{13}, \eta_{13}]^{[1,1,3,3,3,3]} \twoheadrightarrow \mathcal{M}(13, \langle 3 \rangle).$$

Note  $C_{13} = f_{\chi_{13}^3} \overline{f_{\chi_{13}^3}}$ .

4.14. **the case**  $N = 16$ . Put

$$\gamma_{16} = \frac{1}{2}(\alpha_8^{\langle 2 \rangle} - \beta_8),$$

then we get

$$\mathbb{C}[f_{\rho_4}^{\langle 4 \rangle}, \alpha_8, \alpha_8^{\langle 2 \rangle}, \gamma_{16}, \beta_8^{\langle 2 \rangle}]^{[1,1,1,1,1]} \twoheadrightarrow \mathcal{M}(16, \langle 9 \rangle).$$

Next, put

$$\begin{aligned} F_{16} &= \frac{1}{2}(f_{\chi_{16}} + \overline{f_{\chi_{16}}}) = 1 + \sum_{n \in \mathbb{N}} (\sigma_0 * (\text{Re} + \text{Im})\chi_{16})(n)q^n, \\ G_{16} &= \frac{1^{1/4}}{2}(f_{\chi_{16}} - \overline{f_{\chi_{16}}}) = \sum_{n \in \mathbb{N}} (\sigma_0 * (\text{Re} - \text{Im})\chi_{16})(n)q^n, \end{aligned}$$

then we get

$$\&\mathcal{M}_k(16, \rho_4^{k+1}\chi_{16}) = \mathbb{C}f_{\rho_4}^k F_{16} \oplus \mathcal{M}_{k-1}(16, \langle 9 \rangle)G_{16},$$

and

$$\mathcal{M}(16, \langle 9 \rangle)[F_{16}, G_{16}]^{[1,1]} \twoheadrightarrow \mathcal{M}(16, 1).$$

4.15. **the case**  $N = 18$ . We get

$$\mathbb{C}[f_{\rho_3}^{\langle 3 \rangle}, \alpha_9, \alpha_9^{\langle 2 \rangle}, \alpha_6^{\langle 3 \rangle}]^{[1,1,1,1]} \twoheadrightarrow \mathcal{M}(18, \langle 7 \rangle).$$

## 5. HALF INTEGER WEIGHT MODULAR FORMS

**5.1. definitions.** We treat the half-integer cases. For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$  and  $z \in \mathcal{H}$ , we put  $J(\gamma, z) = (\frac{c}{d})(cz + d)^{1/2}$ , where  $(\frac{c}{d})$  is the Kronecker symbol and  $x^{1/2}$  is the "principal" determination of the square root of  $x$ , i.e. the one whose real part is  $> 0$ . For each  $\kappa \in \frac{1}{2}\mathbb{M}$ , we define the weight  $\kappa$  action  $4, 1 \curvearrowright \mathcal{O}(\mathcal{H})$  by  $f|_{\kappa}\gamma(z) = J(\gamma, z)^{-2\kappa}f(\gamma z)$ . If  $\kappa \in \mathbb{M}$ , the definition is consistent with the first one.

For  $\kappa \in \frac{1}{2}\mathbb{M}$  and  $\Gamma \subset \Gamma(4, 1)$  such that  $\Gamma(N, 1) \subset \Gamma$  for some  $N$ , we may define  $\mathcal{M}_{\kappa}(\Gamma)$  too and

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(\Gamma) = \bigoplus_{\kappa \in \frac{1}{2}\mathbb{M}} \mathcal{M}_{\kappa}(\Gamma).$$

For a function (not character)  $\rho : \mathbb{Z}/N^{\times} \rightarrow \mathbb{C}^{\times}$ , we may also define  $\mathcal{M}_{\kappa}(N; \rho)$ .

**Lemma 2.** *If  $\mathcal{M}_{2\kappa}(\Gamma) \cap \mathbb{C}[[q]]q^d = \{0\}$ , then we see*

$$\dim(\mathcal{M}_{\kappa}(\Gamma)) \leq \lfloor \frac{d+1}{2} \rfloor.$$

*Proof.*

$$LHS = \#\{i \mid \mathcal{M}_{\kappa}(\Gamma) \cap (q^i + \mathbb{C}[[q]]q^{i+1}) \neq \emptyset\} \leq \#\{0, 1, \dots, \lfloor \frac{d-1}{2} \rfloor\}$$

□

**5.2. the case  $N = 4$ .** Put

$$\vartheta = \sum_{n \in \mathbb{Z}} q^{n^2},$$

then we show  $\vartheta \in \mathcal{M}_{\frac{1}{2}}(4; \rho_4^{-1/2})$ .

*Proof.* We see  $\vartheta(\gamma z) = J(\gamma, z)\rho_4(d)^{-1/2}\vartheta(z)$  for  $\gamma \in \Gamma_0(4)$  and  $z \in \mathcal{H}$  (cf. [4, p241] or [6]), i.e.  $\vartheta|_{\frac{1}{2}}\gamma = \rho_4(d)^{-1/2}\vartheta$ . The holomorphy of  $\vartheta$  follows from [2, Proposition 1.2.4]. □

We see

$$\mathcal{M}_{k+\frac{1}{2}}(4, 1) = \mathcal{M}_k(4, 1)\vartheta,$$

since  $\supset$  by the above assertion, and  $\dim \mathcal{M}_{k+\frac{1}{2}}(4, 1) \leq \lfloor \frac{(k+1)+1}{2} \rfloor$  by Lemma 2. Note  $f_{\rho_4} = \vartheta^2$ , and we get

$$\mathbb{C}[\vartheta, \alpha_4]^{[\frac{1}{2}, 2]} \twoheadrightarrow \mathcal{M}_{\frac{1}{2}\mathbb{M}}(4, 1).$$

**5.3. the case  $N = 8$ .** We show  $\vartheta^{(2)} \in \mathcal{M}_{\frac{1}{2}}(8; \rho_8\rho_4^{1/2})$ .

*Proof.* For  $h \in \mathbb{N}$  and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4h)$ , with  $\gamma' = \gamma \triangleleft \begin{pmatrix} 1/h & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bh \\ c/h & d \end{pmatrix}$ , we note

$$\begin{aligned} (\vartheta^{(h)}|_{\frac{1}{2}}\gamma)(z) &= J(\gamma, z)^{-1}\vartheta^{(h)}(\gamma z) \\ &= J(\gamma, z)^{-1}\vartheta(\gamma'(hz)) \\ &= J(\gamma, z)^{-1}J(\gamma', hz)\rho_4(d)^{-1/2}\vartheta(hz) \\ &= (\frac{c}{d})(\frac{c/h}{d})\rho_4(d)^{-1/2}\vartheta(hz) \\ &= (\frac{h}{d})\rho_4(d)^{-1/2}\vartheta^{(h)}(z). \end{aligned}$$

We get the assertion from  $(\frac{2}{p}) = \rho_4\rho_8(p)$  for a prime  $p \neq 2$ . □

We get

$$\mathbb{C}[\vartheta, \vartheta_{\langle 2 \rangle}]^{[\frac{1}{2}, \frac{1}{2}]} \twoheadrightarrow \mathcal{M}_{\frac{1}{2}\mathbb{M}}(8, 1).$$

Note  $f_{\rho_8} = \vartheta \vartheta^{\langle 2 \rangle}$ .

**5.4. the case  $N = 12$ .** We show

$$\mathcal{M}_{k+\frac{1}{2}}(12, 1) = \mathbb{C}\vartheta^{2k+1} \oplus \mathcal{M}_k(12, 1)\vartheta_{\langle 3 \rangle}.$$

*Proof.* We see  $\vartheta^{\langle 3 \rangle} \in \mathcal{M}_{\frac{1}{2}}(12; \rho_3 \rho_4^{1/2})$  since  $(\frac{3}{p}) = \rho_3 \rho_4(p)$  for a prime  $p \neq 2, 3$ . Thus we get  $\supset$ , on the other hand we see

$$\dim \mathcal{M}_{k+\frac{1}{2}}(12, 1) \leq 4k + 3.$$

If  $\leq$  is  $=$ , then there is a form  $x \in \mathcal{M}_{k+\frac{1}{2}}(12, 1) \cap (q^{4k+2} + \mathbb{C}[[q]]q^{4k+3})$ . It has to satisfy

$$\begin{aligned} x^2 &\in \mathcal{M}_{2k+1}(12, 1) \cap (q^{8k+4} + \mathbb{C}[[q]]q^{8k+5}) = \{\delta_{12}^{2k+1}\}, \\ \delta_{12}^k \vartheta_{\langle 3 \rangle} x &\in \mathcal{M}_{2k+1}(12, 1) \cap (q^{8k+3} + \mathbb{C}[[q]]q^{8k+4}) = \delta_{12}^{2k}(\gamma_{12} + \mathbb{C}\delta_{12}). \end{aligned}$$

We get contradiction from

$$\begin{aligned} x/\delta_{12}^k &\in q^2 - q^3 - \frac{1}{2}q^4 + \mathbb{C}[[q]]q^5, \\ \vartheta_{\langle 3 \rangle} x/\delta_{12}^k &\in \gamma_{12} + \mathbb{C}\delta_{12}. \end{aligned}$$

□

Note  $\vartheta \vartheta^{\langle 3 \rangle} = f_{\rho_3}^{(4)} + 2\alpha_{12}$ , and we get

$$\mathbb{C}[\vartheta, \vartheta_{\langle 3 \rangle}, \gamma_{12}, \delta_{12}]^{[\frac{1}{2}, \frac{1}{2}, 1, 1]} \twoheadrightarrow \mathcal{M}_{\frac{1}{2}\mathbb{M}}(12, 1).$$

**5.5. the case  $N = 16$ .** We see  $\vartheta^{\langle 4 \rangle} \in \mathcal{M}_{\frac{1}{2}}(16; \rho_4^{-1/2})$  since  $(\frac{4}{d}) = 1$  for  $d \in 4\mathbb{Z} \pm 1$ , and get

$$\mathbb{C}[\vartheta, \vartheta_{\langle 4 \rangle}, \vartheta_{\langle 2 \rangle}^{\langle 2 \rangle}]^{[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]} \twoheadrightarrow \mathcal{M}_{\frac{1}{2}\mathbb{M}}(16, \langle 9 \rangle).$$

Note

$$\begin{aligned} &\vartheta^2 - 4\vartheta^{\langle 4 \rangle} \vartheta_{\langle 4 \rangle} - \vartheta^{\langle 2 \rangle 2} \\ &= \sum_{m, n \in \mathbb{Z}} q^{m^2+n^2} - 2 \sum_{m \in 2\mathbb{Z}, n \in 2\mathbb{Z}+1} q^{m^2+n^2} - \sum_{m, n \in \mathbb{Z}} q^{2(m^2+n^2)} \\ &= \sum_{m, n \in \mathbb{Z}} q^{(2m)^2+(2n)^2} + \sum_{m, n \in \mathbb{Z}} q^{(2m+1)^2+(2n+1)^2} - \sum_{m, n \in \mathbb{Z}} q^{2(m^2+n^2)} \\ &= \sum_{m, n \in \mathbb{Z}} q^{2((m+n)^2+(m-n)^2)} + \sum_{m, n \in \mathbb{Z}} q^{2((m+n+1)^2+(m-n)^2)} - \sum_{m, n \in \mathbb{Z}} q^{2(m^2+n^2)} \\ &= 0, \end{aligned}$$

and thus

$$\vartheta \vartheta^{\langle 4 \rangle} = \vartheta^{\langle 4 \rangle 2} + 2\vartheta_{\langle 4 \rangle} \vartheta^{\langle 4 \rangle} = \vartheta^{\langle 4 \rangle 2} + \frac{1}{2}(\vartheta^2 - \vartheta^{\langle 2 \rangle 2}) = f_{\rho_4}^2 + 2\alpha_8.$$

## 6. HILBERT FUNCTION

For a graded-ring  $R = \bigoplus_{k \in \mathbb{M}} R_k$  satisfying  $\dim R_k < \infty$  for all  $k \in \mathbb{M}$ , put

$$\text{Dim}(R) = \sum_{k \in \mathbb{M}} (\dim R_k) t^k \in \mathbb{Z}[[t]].$$

We see  $\text{Dim}(\mathbb{C}) = 1$ .

**Lemma 3.** *Let  $R = \bigoplus_{k \in \mathbb{M}} R_k$  and  $n \in \mathbb{N}$ , then we see*

$$\text{Dim}(R[X]^{[n]}) = \frac{\text{Dim}(R)}{1 - t^n}.$$

*Proof.* From

$$(R[X]^{[n]})_k = R_k \oplus R_{k-n}X \oplus R_{k-2n}X^2 \oplus \cdots.$$

□

For example,

$$\text{Dim}(\mathbb{C}[X, Y]^{[1, n]}) = \frac{1}{(1-t)(1-t^n)} = \sum_{k \in \mathbb{M}} \left( \left\lfloor \frac{k}{n} \right\rfloor + 1 \right) t^k.$$

Next, let  $R$  be a ring and  $n \in \mathbb{N}$ . If  $O_i \in XZ_i + R[Z_1, \dots, Z_n]$  ( $i = 1, \dots, n$ ), then we see

$$\begin{aligned} R[Z_1, \dots, Z_n]X &= (R + R[Z_1, \dots, Z_n]Z_1 + \cdots + R[Z_1, \dots, Z_n]Z_n)X \\ &\subset (O_1, \dots, O_n) + RX + R[Z_1, \dots, Z_n], \end{aligned}$$

and  $R[Z_1, \dots, Z_n]X^i \subset (O_1, \dots, O_n) + RX^i + R[Z_1, \dots, Z_n]$  by induction  $i$ , thus

$$R[X, Z_1, \dots, Z_n] = (O) + R[X] + R[Z_1, \dots, Z_n].$$

**Lemma 4.** *For  $n \in \mathbb{N}$ , we see*

$$\text{Dim}(\mathbb{C}[X_0, X_1, \dots, X_{n+1}]^{[1, 1, \dots, 1]} / \mathfrak{a}) = \frac{1 + (n-1)t}{(1-t)^2} = \sum_{k \in \mathbb{M}} (nk + 1)t^k,$$

where

$$\mathfrak{a} = (X_i X_{j+1} - X_{[\frac{i+j+1}{2}]} X_{[\frac{i+j+2}{2}]} \mid 0 \leq i < j < n).$$

*Proof.* By induction on  $n$ . The assertion is trivial if  $n = 1$ , and by the above criterion we see

$$\begin{aligned} \mathbb{C}[X_0, X_1, \dots, X_{n+1}] &= (X_0 X_2 - X_1 X_1) + \cdots + (X_0 X_{n+1} - X_{[\frac{n+1}{2}]} X_{[\frac{n+2}{2}]}) \\ &\quad + \mathbb{C}[X_1][X_0] + \mathbb{C}[X_1][X_2, \dots, X_n]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{C}[X_0, X_1, \dots, X_n] &= \mathfrak{a} + \mathbb{C}[X_0, X_1] + \mathbb{C}[X_1, X_2] + \cdots + \mathbb{C}[X_{n-1}, X_n] \\ &= \mathfrak{a} \oplus \mathbb{C}[X_0, X_1] \oplus \mathbb{C}[X_1, X_2]X_2 \oplus \cdots \oplus \mathbb{C}[X_{n-1}, X_n]X_n. \end{aligned}$$

□

**Lemma 5.** *We see*

$$\dim(\mathbb{C}[X_0, X_1, Y_0, Y_1]^{[1,1,2,2]}/\mathfrak{a}) = \sum_{k \in \mathbb{M}} (k + \lfloor \frac{k}{2} \rfloor + 1)t^k,$$

where

$$\mathfrak{a} = (X_1^3 - X_2Y_1, X_2^3 - X_1Y_2, X_1^2X_2^2 - Y_1Y_2).$$

*Proof.* We have

$$\begin{aligned} \mathbb{C}[X_0, X_1, Y_1, Y_2] &= (X_1^2X_2^2 - Y_1Y_2) + \mathbb{C}[X_1, X_2][Y_1] + \mathbb{C}[X_1, X_2][Y_2] \\ &= \mathfrak{a} \oplus \mathbb{C}[X_0, X_1] \oplus \mathbb{C}[X_1, Y_1]Y_1 \oplus \mathbb{C}[X_2, Y_2]Y_2. \end{aligned}$$

□

## 7. STRUCTURE

7.1. **the case**  $N = 1$ . We see

$$\begin{aligned} \dim(\mathbb{C}[X, Y]^{[2,3]}) &= \frac{1}{t^2} \left( \frac{1}{(1-t)(1-t^2)} - \frac{1}{(1-t)(1-t^3)} \right) \\ &= \sum_{k \in \mathbb{N}} \left( \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{3} \right\rfloor \right) t^{k-2} \\ &= \sum_{k \in \mathbb{N}} \dim \mathcal{M}_{2k}(1) t^k, \end{aligned}$$

thus

$$\mathcal{M}(1) \simeq \mathbb{C}[E_4, E_6]^{[4,6]}.$$

7.2. **the case**  $N = 2$ . We see

$$\mathcal{M}(2) \simeq \mathbb{C}[C_2, \alpha_2]^{[2,4]}.$$

7.3. **the case**  $N = 3$ . We see

$$\mathcal{M}(3, 1) \simeq \mathbb{C}[f_{\rho_3}, g_{3;\rho_3}]^{[1,3]}.$$

7.4. **the case**  $N = 4$ . We see

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(4, 1) \simeq \mathbb{C}[\vartheta, \alpha_4]^{[\frac{1}{2}, 2]}.$$

7.5. **the case**  $N = 5$ . We see

$$\mathcal{M}(5, 1) \simeq \mathbb{C}[f_{\chi_5}, \overline{f_{\chi_5}}]^{[1,1]}.$$

7.6. **the case**  $N = 6$ . We see

$$\mathcal{M}(6, 1) \simeq \mathbb{C}[f_{\rho_3}, \alpha_6]^{[1,1]}.$$

7.7. **the case**  $N = 7$ . We see

$$\mathcal{M}(7, 1) \simeq \mathbb{C}[f_{\rho_7}, f_{\chi_7}, \overline{f_{\chi_7}}]^{[1,1,1]} / (f_{\rho_7}^2 - f_{\chi_7} \overline{f_{\chi_7}}).$$

7.8. **the case**  $N = 8$ . We see

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(8, 1) \simeq \mathbb{C}[\vartheta, \vartheta^{(2)}]^{[\frac{1}{2}, \frac{1}{2}]}.$$

7.9. **the case**  $N = 9$ . Put

$$u_9 = f_{\rho_3}^{(3)} + 3 \cdot 1^{1/6} \alpha_9,$$

$$O_9 = f_{\chi_9}^2 - \overline{u_9 f_{\chi_9}},$$

$$O'_9 = f_{\chi_9} \overline{f_{\chi_9}} - u_9 \overline{u_9}$$

then we see  $O_9 \in \mathcal{M}_2(9, \overline{\chi_9}) \cap \mathbb{C}[[q]]q^2 = \{0\}$ ,

$$O'_9 = f_{\chi_9} \overline{f_{\chi_9}} - \frac{f_{\chi_9}^2 \overline{f_{\chi_9}}^2}{f_{\chi_9} \overline{f_{\chi_9}}} = 0,$$

and

$$\mathcal{M}(9, 1) \simeq \mathbb{C}[u_9, \overline{u_9}, f_{\chi_9}, \overline{f_{\chi_9}}]^{[1,1,1,1]} / (O_9, \overline{O_9}, O'_9).$$



7.10. **the case  $N = 10$ .** Put

$$\begin{aligned} u_{10} &= f_{\chi_5}^{(2)} + (1 - 1^{1/4})f_{\chi_5 \langle 2 \rangle}, \\ v_{10} &= f_{\chi_5}^{(2)} + (1 + 3 \cdot 1^{1/4})f_{\chi_5 \langle 2 \rangle}, \\ O_{10} &= u_{10}^2 - \overline{u_{10}v_{10}}, \\ O'_{10} &= u_{10}\overline{u_{10}} - v_{10}\overline{v_{10}} \end{aligned}$$

then we see

$$\mathcal{M}(10, 1) \simeq \mathbb{C}[u_{10}, v_{10}, \overline{u_{10}}, \overline{v_{10}}]^{[1, 1, 1, 1]} / (O_{10}, \overline{O_{10}}, O'_{10}).$$

7.11. **the case  $N = 11$ .**

$$\mathcal{M}(11, 1) \simeq \mathbb{C}[f_{\chi_{11}}, f_{\chi_{11}^3}, f_{\rho_{11}}, \overline{f_{\chi_{11}^3}}, \overline{f_{\chi_{13}}}]^{[1, 1, 1, 1, 1]} / ?.$$

7.12. **the case  $N = 12$ .** Put

$$\begin{aligned} u_{12} &= f_{\rho_3}^{(4)} + 6\alpha_{12} + 12\alpha_6^{(2)}, \\ v_{12} &= f_{\rho_3}^{(4)} - 2\alpha_{12} + 4\alpha_6^{(2)}, \end{aligned}$$

Put

$$\begin{aligned} O_{12a} &= \vartheta^3 - \vartheta^{(3)}u_{12}, \\ O_{12b} &= \vartheta^{(3)3} - \vartheta v_{12}, \\ O_{12c} &= \vartheta^2 \vartheta^{(3)2} - u_{12}v_{12}, \end{aligned}$$

then we see

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(12, 1) \simeq \mathbb{C}[\vartheta, \vartheta^{(3)}, u_{12}, v_{12}]^{[\frac{1}{2}, \frac{1}{2}, 1, 1]} / (O_{12a}, O_{12b}, O_{12c}).$$

7.13. **the case  $N = 13$ .**

$$\mathcal{M}(13, \langle 3 \rangle) \simeq \mathbb{C}[f_{\chi_{13}^3}, \overline{f_{\chi_{13}^3}}, f_{3; \chi_{13}^3}, \overline{f_{3; \chi_{13}^3}}, g_{3; \chi_{13}^3}, \overline{g_{3; \chi_{13}^3}}]^{[1, 1, 3, 3, 3, 3]} / ?.$$

7.14. **the case  $N = 16$ .** Put

$$\begin{aligned} u_{16} &= \vartheta^{(4)} + 2 \cdot 1^{1/4} \vartheta_{\langle 4 \rangle}, \\ v_{16} &= \vartheta^{(4)} - 2 \cdot 1^{1/4} \vartheta_{\langle 4 \rangle}, \end{aligned}$$

then we see

$$\mathcal{M}_{\frac{1}{2}\mathbb{M}}(16, \langle 9 \rangle) = \mathbb{C}[u_{16}, v_{16}, \vartheta^{(2)}]^{[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]} / (\vartheta^{(2)2} - u_{16}v_{16}).$$

7.15. **the case  $N = 18$ .** we put

$$\begin{aligned} u_6 &= f_{\rho_3} - 4\alpha_6 \\ O_{18a} &= \alpha_9^2 - u_6^{(3)}\alpha_9^{(2)}, \\ O_{18b} &= \alpha_9\alpha_9^{(2)} - u_6^{(3)}\alpha_6^{(3)}, \\ O_{18c} &= \alpha_9^{(2)2} - \alpha_9\alpha_6^{(3)}, \end{aligned}$$

then we see  $O_{18b}, O_{18c} \in \mathcal{M}_2(18, \langle 7 \rangle) \cap q^7 = \{0\}$ , thus

$$\mathcal{M}(18, \langle 7 \rangle) \simeq \mathbb{C}[u_6^{(3)}, \alpha_9, \alpha_9^{(2)}, \alpha_6^{(3)}]^{[1, 1, 1, 1]} / (O_{18a}, O_{18b}, O_{18c}).$$

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